When the Chips are Down...Understanding Arises

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Flaws of the Number Line Model

The use of a well designed and intuitive model can greatly help to facilitate understanding of any new concept in mathematics. This is especially true when first presenting the concepts of integers and the arithmetic of integers. Realizing this, most elementary and middle school classrooms prominently feature a large scale representation of a number line on at least one wall and many teachers utilize these as either their primary or exclusive model for integer instruction. Unfortunately, while the number line is an excellent model for use in ordering integers or instructing arithmetic of whole numbers, its utility is weakened with arithmetic of integers. For example, \(4 + 2\) and \(5 - 3\) are easily modeled as follows on number lines.

\[
\begin{align*}
4 & \quad + \quad 2 \quad = \quad 6 \\
\text{-4} & \quad \text{-3} & \quad \text{-2} & \quad \text{-1} & \quad 0 & \quad 1 & \quad 2 & \quad 3 & \quad 4 & \quad 5 & \quad 6
\end{align*}
\]

\[
\begin{align*}
5 & \quad - \quad 3 \quad = \quad 2 \\
\text{-6} & \quad \text{-5} & \quad \text{-4} & \quad \text{-3} & \quad \text{-2} & \quad \text{-1} & \quad 0 & \quad 1 & \quad 2 & \quad 3 & \quad 4 & \quad 5 & \quad 6
\end{align*}
\]

Clearly, addition means to include another arrow going to the right to increase the answer and subtraction means include an arrow going to the left to decrease the answer. Right? But wait, to include integers, \(-4\) must mean an arrow pointing to the left since we always start at zero and \(-4\) could be thought of as \(0 - 4\)…like this.

\[
\begin{align*}
\text{-4} \\
\text{-6} & \quad \text{-5} & \quad \text{-4} & \quad \text{-3} & \quad \text{-2} & \quad \text{-1} & \quad 0 & \quad 1 & \quad 2 & \quad 3 & \quad 4 & \quad 5 & \quad 6
\end{align*}
\]

But then how do we deal with \(-3 + (-2)\) ? Since addition means to include another arrow going to the right to increase the answer, we put an arrow to the right. No wait… subtraction means include an arrow going to the left to decrease the answer, and we see a minus sign, so we put the arrow to the left. At this point, the only thing that seems clear or intuitive is that the model is not. Can we resolve the number line’s issue with integers? Sure, teachers have done so for years. To do so, however, requires us to resort to bizarre ideas like negative means to “turn around” or addition is “arrow head to arrow tail” and subtraction is “arrow head to arrow head”. Needless to say, the intuitiveness and clarity of the model is lost in the new “rules” we have to create to allow it to properly function. Instead of all these convoluted rules, perhaps we could just consider trying a different model, one more naturally disposed to integer operations – the chip model.
The Manipulatives of the Chip Model

The chip model uses very simple manipulative materials, two-color counters. In this model, integers are understood to have two characteristic components: sign and quantity. Sign is represented by color. Accordingly, the color from one side of the counter is arbitrarily declared to represent positive and the other is negative. For the purposes of this discussion, let’s say red (shaded grey in this document) is positive and white is negative.

Positive

Negative

Quantity is intuitively represented by the number of chips grouped together. For example, the integer \(-4\) is depicted as four white chips grouped together. Similarly, the integer 7 would be depicted as seven red chips grouped together.

\[
\begin{array}{c}
\text{- 4} \\
\end{array}
\quad
\begin{array}{c}
\text{7}
\end{array}
\]

Shape or structure in the groupings is irrelevant, but students will frequently create arrays or symmetries. Discourage students from stacking chips, however, as this does not allow for the quantity of chips to be quickly observed and counted. Furthermore, the chip model allows for a natural and intuitive introduction to the more advanced concept of absolute value as absolute value can easily be interpreted as the quantity of chips in the representation of an integer, disregarding color.

Addition with the Chip Model

Addition using the chip model is based on students’ intuitive understanding that addition means to group together. Thusly, when adding like types of integers, we merely combine groupings and count the total number of chips in the final grouping. Observe the following example in which we model \(3 + 5\).

\[
\begin{array}{c}
\text{3} \\
\text{+} \\
\text{5}
\end{array}
\quad
\rightarrow
\quad
\begin{array}{c}
\text{=} \\
\text{8}
\end{array}
\]
The process is similarly intuitive with negative integers as the following example, in which we model \((-4) + (-3)\), demonstrates.

\[ (-4) + (-3) = -7 \]

When adding positive and negative integers, we must instruct one more aspect of the model. When a positive and a negative chip are placed together, they cancel each other out on a one to one basis and are removed from the workspace. For example, consider the model of \(2 + (-2)\).

\[ 2 + (-2) = 0 \]

Many different analogies can be provided to students to help them accept this fact. If your group of students is scientifically inclined, you could refer to the positive and negative chips as matter and antimatter since when these are combined they annihilate each other. Due to the popularity of the Harry Potter books and movies, I have also described positive chips as Harry Potter and negative chips as his invisibility cloak. When Harry Potter and his invisibility cloak are apart, they are both visible. I verify this with the students by having them describe what both look like…Harry Potter has glasses and a scar…the invisibility cloak is shiny and silver in color. When the two are combined, however, they disappear. Infinitely many other analogies are possible, but it is highly recommended that you keep your references current for maximum student acceptance and credibility.

To add positive and negative integers is a four step process.

1.) Represent the integers to be added with groupings of appropriately colored chips.
2.) Combine the groupings of chips, but pair positive and negative chips (red and white chips) on a one to one basis as possible.
3.) Remove each positive and negative pairing from the workspace.
4.) Count the remaining chips and note the color to determine whether the final result is positive or negative.

As an example, let us model \(4 + (-7)\).
Thusly, when adding integers, students clearly see the color, i.e., positivity or negativity, of the final result will match whichever color had the most chips. Or more technically phrased, the sign of the sum will be that of the summand whose absolute value is the greatest.

**Subtraction with the Chip Model**

Performing subtraction using the chip model is again designed to build on the students’ most concrete foundations of the operation, i.e., that *subtraction means to take away*. Accordingly, when subtracting with the chip model, we simply remove the subtracted quantity of chips from the working area. For example, we demonstrate $6 - 2$ with the model.

We note that when working positives or negatives exclusively, as long as the subtrahend, i.e., the quantity to be removed, is less than the minuend, i.e., the starting quantity, the difference retains the sign of the original integers…and the model is quite simple. To further illustrate this point, we model $(-5) - (-3)$. 

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The natural question that occurs at this point is: “What if we need to take away more than we have on the working area?” Consider the problem $6 - 9$.

Even if we take away all 6 chips…we still are 3 chips short of taking away 9. How do we remove the last three chips?

To deal with this situation, i.e., when the absolute value of the subtrahend is greater than the absolute value of the minuend, we need to slightly modify our understanding of what happens when positives and negatives are paired. Instead of thinking of these pairings as eliminating the chips, we need to think the pairings as causing the chips to become invisible or unseen. In other words, $a + (-a) = 0$, and, thusly, since 0 is represented by blank space in the model, we can’t see the result. Hence, if we need to subtract more chips than we have on the working area, we must “allow” pairs of positives and negatives to become visible so that we can remove the additional chips to complete the subtraction. The earlier Harry Potter and the invisibility cloak analogy is particularly helpful here as we can say that an unseen Harry is wearing his cloak and watching the other chips. He never really left; he is just invisible and can reappear at any time by simply taking off his cloak.

Consider again the problem $6 - 9$. Start with the 6 positive chips. Since we need 3 more positive chips, add to the model 3 positive-negative pairings. This does not change the value from 6.
Now if we attempt the problem again, we can easily regroup and remove 9 positive chips.

\[
\begin{array}{ccc}
6 & - & 9 \\
\hline
\end{array}
\Rightarrow \begin{array}{c}
-3
\end{array}
\]

It does not matter if we start with positives or negatives, the process still works. To illustrate this, consider now the modeling of \(-3 - (-7)\).

\[
\begin{array}{ccc}
(-3) & - & (-7) \\
\hline
\end{array}
\Rightarrow \begin{array}{c}
4
\end{array}
\]

The model even works for situations where we don’t have any chips shown of the color we are trying to subtract. An example of this would be \(-4 - 2\).

\[
\begin{array}{ccc}
(-4) & - & 2 \\
\hline
\end{array}
\Rightarrow \begin{array}{c}
-6
\end{array}
\]
Multiplication with the Chip Model

Once the modeling of addition and subtraction are mastered, the chip model quickly extends into multiplication. Remind students that multiplication can be thought of as repeated addition of many groupings where the first factor of the multiplication problem establishes the number of groupings and the second factor tells us the quantity and type of chip in each grouping. For example, $3 \times 5$ means 3 groupings of 5 positive chips. Similarly, $2 \times (-4)$ means 2 groupings of 4 negative chips. These are modeled as follow.

If the first factor is negative, inform the students that this means to “take away” the groupings. To do this we will again need to use the “invisible pairings”. This is demonstrated in the modeling of $(-3) \times 2$, which requires us to take away 3 groupings of 2 positives. Note, before we take away…we start with nothing, i.e., 0. Hence, the initially blank work area.
Division with the Chip Model

The chip model can be further extended to include division. First, represent the dividend, i.e., value to be divided, with a single grouping of appropriately colored chips. Next, recognize that the divisor, i.e., the value doing the dividing, is telling us the number of groupings into which we want to partition the original value. Proceed to partition the chips as required by the problem and the quotient will be the number and color of the chips in each partition. As an example, we model \((-12) \div 4\).

\[
\begin{align*}
\text{(- 12)} & \rightarrow \quad \text{\textcolor{red}{\bigcirc} \quad \textcolor{red}{\bigcirc} \quad \textcolor{red}{\bigcirc} \quad \textcolor{red}{\bigcirc} } \\
\div & \quad 4 \quad = \quad \textcolor{red}{\bigcirc} \quad \textcolor{red}{\bigcirc} \quad \textcolor{red}{\bigcirc} \quad \textcolor{red}{\bigcirc} \\
\end{align*}
\]

Since there are three negative chips in each grouping, the quotient is \(-3\). Since partitioning is the opposite process of grouping together, students should be able to quickly realize that division and multiplication are inverse operations, i.e., the reverses of each other. Furthermore, once recognition of the connection between multiplication and division is established, the need for modeling excessive quantities of division problems is eliminated. This is particularly useful as modeling division with a negative divisor is somewhat problematic. The process can be modeled with extensions of the “take away” principle, but tends to be somewhat confusing and, thusly, negates the purpose of using a model, i.e., to improve student understanding and decrease confusion. As such, it is frequently easier to omit using the model in such cases and instead rely on standard algorithms, multiplication fact families, and generalizations of the behavior of integers in arithmetic operations.

Generalizations with the Chip Model

From a very brief number of examples, students should also be able to start forming conjectures as to the nature of integer arithmetic. These generalizations should include the standard “rules” we teach students about performing integer arithmetic and are easily tested with the model. Examples should include:

- A negative plus a negative is a negative
- A positive plus a positive is a positive
- A positive times a positive is a positive
- A positive times a negative is a negative
- A negative times a positive is a negative
- A negative times a negative is a positive

Once students understand and accept these generalizations, the model has served its purpose and is no longer needed, i.e., students are fully prepared to utilize standard, more efficient algorithms.
§111.23. Mathematics, Grade 7.

(b) Knowledge and skills.

(1) Number, operation, and quantitative reasoning. The student represents and uses numbers in a variety of equivalent forms. The student is expected to:

(A) compare and order integers and positive rational numbers;

(2) Number, operation, and quantitative reasoning. The student adds, subtracts, multiplies, or divides to solve problems and justify solutions. The student is expected to:

(C) use models, such as concrete objects, pictorial models, and number lines, to add, subtract, multiply, and divide integers and connect the actions to algorithms;

Materials and Pricing Information

The only materials required to implement this activity into a classroom are a set of two-color counters. The two-color counters used in this presentation were obtained from EAI Education (http://www.eaieducation.com/) at a cost of $17.95 for a jar of 1000 counters. Two-color counters, however, are quite common and should available at a myriad of other local and internet sources for comparable pricing.