Money, Mayans, and Manipulatives

Explorations in Numeration

ME² by the Sea
June 13, 2008

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The Monetary Connection to Ancient Numeration

(You aren’t going to get rich here…it’s pedagogical not compensational)

Egyptian

The Egyptian numeration system is a purely additive system and, as such, is one of the easiest systems to learn. The basic Egyptian system consists of seven hieroglyphic symbols, each representing a successive power of ten. The symbols and their equivalent value are as follows:

- : = 1
- : = 10
- : = 100
- : = 1,000
- : = 10,000
- : = 100,000
- : = 1,000,000

Translating Hindu-Arabic numbers into the Egyptian numeration system is performed most efficiently by first writing Hindu-Arabic numbers in expanded notation. For example,

\[ 325,806 = (3 \times 100,000) + (2 \times 10,000) + (5 \times 1,000) + (8 \times 100) + (0 \times 10) + (6 \times 1) \]

Once accomplished, we merely replace each parenthetical expression with an equivalent number of the appropriate Egyptian symbols.

\[ 325,806 = \]

Order in the Egyptian system is irrelevant. All that matters is the multiplicity and the value of the symbols used. This independence of order allows for very intuitive and accessible connections to monetary systems. Specifically, one can think of each Egyptian symbol as being analogous to a “dollar bill” of equivalent value and, thusly, converting values into Egyptian numeration becomes comparable to the process of making change. For example, suppose you were a cashier at a retail store and needed to give a customer $84 of change. How would you do this most efficiently? You would want use the largest bills first and calculate how much more “change” would have to be given. Worked out fully, the process would appear as follows.

\[ $84 = (1 \times $50) + $34 \]
\[ = (1 \times $50) + (1 \times $20) + $14 \]
\[ = (1 \times $50) + (1 \times $20) + (1 \times $10) + $4 \]
\[ = (1 \times $50) + (1 \times $20) + (1 \times $10) + (0 \times $5) + (4 \times $1) \]
Similarly, a Hindu-Arabic to Egyptian conversion could look like the following.

\[
3,217 = \begin{array}{c}
\text{dots} \\
\text{bars} \\
\text{squares}
\end{array} + 217
\]

\[
= \begin{array}{c}
\text{dots} \\
\text{bars} \\
\text{squares}
\end{array} + 17
\]

\[
= \begin{array}{c}
\text{dots} \\
\text{bars} \\
\text{squares}
\end{array} + 7
\]

\[
= \begin{array}{c}
\text{dots} \\
\text{bars} \\
\text{squares}
\end{array}
\]

Mayan

The Mayan numeration system is a vertically aligned place value system roughly based on powers of twenty. The place values for the Mayan system, written in ascending order, are the following.

\[
\begin{align*}
1 & \quad 20 & \quad 18 \times 20 & \quad 18 \times 20^2 & \quad 18 \times 20^3 & \quad 18 \times 20^4 & \quad \cdots \\
& & (360) & (7,200) & (144,000) & (2,880,000)
\end{align*}
\]

The diversion from a pure base twenty system in the third place value is attributed to the fact that the Maya primarily used their numeration system for the recording of dates. In their calendar, each month had 20 days and the year consisted of 18 months. Thus, the Mayan place values could be interpreted as representing days, months (units of 20 days), and years (units of 18 months or 18 x 20 days). The remaining 5 days of the 365 day year were unnumbered for religious and superstitious reasons.

Despite using such a complex place value system, the Maya retained great simplicity in their numeration and only utilized three separate symbols. The symbols and their assigned values are:

\[
\begin{align*}
\text{•} &= 1 \\
\text{||} &= 5 \\
\text{≈} &= 0
\end{align*}
\]

The Maya used these symbols to create all values from 0 – 19 by following a general rule of placing the dots above the bars in values requiring the use of both symbols. This is illustrated below.

\[
\begin{align*}
\text{••••} &= 4 \\
\text{•••} &= 7 \\
\text{||} &= 10 \\
\text{≈≈≈} &= 18
\end{align*}
\]

As with Egyptian numeration, conversions to and from the Mayan system are easily instructed by utilizing the connection to money. For example, suppose we wanted to convert 126,014 into Mayan numerals. How would we give the change? First we need to realize what our “cash register drawer” would look like, i.e., recall the place values given above. Clearly, we wouldn’t want to use any 144,000’s as they have too great a value. So we determine how many 7,200’s we would need and calculate the remaining “change”. The entire process would look like this.

\[
126,014 = (17 \times 7,200) + 3,614
\]

\[
= (17 \times 7,200) + (10 \times 360) + 14
\]

\[
= (17 \times 7,200) + (10 \times 360) + (0 \times 20) + (14 \times 1)
\]
Finally, we take the coefficients from each place value (17, 10, 0, 14), write these vertically descending by place value, and then replace the Hindu-Arabic numerals with equivalently valued Mayan symbols.

\[ 17 \quad 10 \quad 0 \quad 14 \]

Converting from Mayan numeration to Hindu-Arabic is accomplished by simply reversing the process and constructing the associated expanded notation.

\[
\begin{align*}
\bullet\bullet\bullet\bullet & \quad 4 \quad 4 \times 7,200 \\
\bullet\bullet\bullet & \quad 13 \quad 13 \times 360 \\
\bullet\bullet & \quad 11 \quad 11 \times 20 \\
\bullet & \quad 6 \quad 6 \times 1
\end{align*}
\]

Hence, the expanded notation is \((4 \times 7,200) + (13 \times 360) + (11 \times 20) + (6 \times 1) = 33,706\).

The “cash drawer” concept of explaining conversion between numeration systems quickly and easily imparts a thorough comprehension of multiple numeration systems and a better understanding of our own Hindu-Arabic base ten system.

The basic details of other numeration systems are now included to allow for further explorations using this method.
Ancient Numeration Menagerie

(Just enough information to make you dangerous…)

Babylonian

The Babylonian numeration system is a horizontally aligned place value system based on powers of sixty. The place values for the Babylonian system, written in ascending order, are the following.

\[
1, 60, 60^2, 60^3, 60^4, \ldots
\]

\[
(3, 600) \quad (216, 000) \quad (12, 960, 000)
\]

The Babylonian system utilizes only two separate cuneiform symbols. The symbols and their assigned values are:

\[\begin{align*}
\blacktriangledown &= 1 \\
\blacktriangleright &= 10
\end{align*}\]

(Note: a third symbol, \(\blacktriangleleft\), representing 0 appeared very late in Babylonian history and some controversy exists as to its introduction, usage, and commonality. Due to this controversy, this symbol is omitted from this discussion.)

The Babylonians used these symbols to create all values from 1 – 59 by following a general rule of placing the “chevrons” (10’s) always to the left of the “wedges” (1’s) in values requiring the use of both symbols. Furthermore, when using multiple chevrons, the Babylonians attempted to align the symbols to roughly form a left pointing triangle. Similarly, when using multiple wedges, the Babylonians attempted to align the symbols to roughly form a downward pointing triangle. This is illustrated below.

\[\begin{align*}
\blacktriangledown &= 6 \\
\blacktriangleright \blacktriangledown &= 30 \\
\blacktriangledown \blacktriangledown &= 24 \\
\blacktriangledown \blacktriangleright \blacktriangledown &= 58
\end{align*}\]

Conversions to and from the Babylonian system are also easily instructed by utilizing the connection to money. For example, suppose we wanted to convert 10,824,092 into Babylonian numerals. How would we give the change? First we need to realize what our “cash register drawer” would look like, i.e., recall the place values given above. Clearly we wouldn’t want to use any 12,960,000’s as they have too great a value. So we determine how many 216,000’s we would need and calculate the remaining “change”. The entire process would look like this.

\[
11,821,092 = (54 \times 216,000) + 157,092
\]

\[
= (54 \times 216,000) + (43 \times 3,600) + 2,292
\]

\[
= (54 \times 216,000) + (43 \times 3,600) + (38 \times 60) + (12 \times 1)
\]

\[
= (54 \times 60^3) + (43 \times 60^2) + (38 \times 60) + (12 \times 1)
\]
Finally, we take the coefficients from each place value (54, 43, 38, 12), write these horizontally from left to right by descending by place value, and then replace the Hindu-Arabic numerals with equivalently valued Babylonian symbols. Note, the Babylonians did not leave spaces between place values, so any such spaces used for our convenience must be removed before the process is complete.

\[
\begin{array}{cccc}
54 & 43 & 38 & 12 \\
\end{array}
\]

\[
\begin{array}{cccc}
\vvvv & \vvv & \vv & \v \\
\vvv & \vv & \v \\
\end{array}
\]

Converting from Babylonian to Hindu-Arabic numeration is accomplished by simply reversing the process and constructing the associated expanded notation. Since no spaces are left in Babylonian numeration, determination of place value separations must be made by following two simple rules:

a.) Within a single place value, chevrons (10’s) must be to the left of wedges (1’s). Thus, if you see wedges to the left of chevrons, a place value change must have occurred. An example is shown below with the place value change illustrated by a dashed line.

\[
\begin{array}{cccc}
\vvvv & \vvv & \vv & \v \\
\vvv & \vv & \v \\
\end{array}
\]

b.) Obvious regroupings of the same type of symbol frequently represent a place value change. Recall that the Babylonians liked to group the repeated symbols into triangular formations. If such a formation is not used, but could have been, there must be a reason – a place value change! An example is shown below with the place value change illustrated by a dashed line.

\[
\begin{array}{cccc}
\vvvv & \vvv & \vv & \v \\
\vvv & \vv & \v \\
\end{array}
\]
This final example demonstrates the entire process by using the techniques to determine place values, constructing the associated expanded notation, and, thereby, converting a number from Babylonian to Hindu-Arabic numeration.

\[
\begin{align*}
(4 \times 60^3) + (53 \times 60^2) + (36 \times 60) + (2 \times 1) &= 1,056,962.
\end{align*}
\]

**Chinese Rod System**

The Chinese Rod system is an alternative set of numeration symbols designed to be used with our standard base ten place values. In this system, a single line or stroke represents one and repeated strokes equate to the values of two through five. To represent a value is larger than five, a perpendicular stroke, denoting five, is placed on the top of the symbol and the number of units above 5 is written as standard strokes below. For example, seven would be a perpendicular five stroke with two normal strokes below. A blank space is used to represent 0. In order to prevent confusion in number like 32, Chinese rod system symbols alternate between vertical and horizontal stroke alignments in successive powers of ten. The symbols and their equivalent values are as follows:

Even Powers of 10, e.g. \(10^0 = 1, 10^2 = 100\)

\[
\begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 0 \\
\hline
\end{array}
\]

Odd Powers of 10, e.g. \(10^1 = 10, 10^3 = 1,000\)

\[
\begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 0 \\
\hline
\end{array}
\]
Translating from Hindu-Arabic into Chinese Rod system numeration is accomplished by merely writing the equivalent symbols in each place value. Care must be given to use the correct set of symbols for each place value. For example,

\[
\begin{align*}
40,318,972 & \quad \equiv \quad \| \mid \equiv \mid \equiv \| \\
5 & \quad 9 & \quad 0 & \quad 2 & \quad 6 & \quad 8 & \quad 1
\end{align*}
\]

Translating from the Chinese Rod system into Hindu Arabic numeration is again accomplished by simply writing the equivalent value for each symbol. For example,

\[
\begin{align*}
\| \| \| \| \| \equiv & \quad \equiv \mid \equiv \mid \equiv \mid \\
5 & \quad 9 & \quad 0 & \quad 2 & \quad 6 & \quad 8 & \quad 1
\end{align*}
\]

5,902,681

**Roman**

The Roman numeration system is a modified additive system in that it incorporates subtractive elements based on position to reduce the number of repeated symbols. The Roman system consists of seven symbols representing a successive powers of ten and certain multiples of five. The symbols and their equivalent values are as follows:

\[
\begin{align*}
I & = 1 & \quad L & = 50 \\
V & = 5 & \quad C & = 100 & \quad M & = 1,000 \\
X & = 10 & \quad D & = 500
\end{align*}
\]

Roman numerals are created using the following rules.

a.) Symbols of smaller value to the right of a symbol of larger value are added together.

For example, \( XI = 10 + 1 = 11 \)

b.) Only symbols representing powers of 10 (I, X, C, M) may be repeated and may only be repeated three times.

For example, \( LXXX = 80 \) is correct, but 65 may not be written as \( LVVV \).

c.) A symbol of smaller value to the left of a symbol of larger value is subtracted from the larger value.

For Example, \( CD = 500 - 100 = 400 \)
d.) Only symbols representing powers of 10 (I, X, C, M) may be subtracted, may only be subtracted once, and may only be subtracted from the next two symbols of greater value. In summary, I can be subtracted once from V or X. X can be subtracted once from L or C. C can be subtracted once from D or M.

e.) Each horizontal line written above a series of symbols represents multiplication by 1000. An easy way to remember this rule is that each line above roman symbols corresponds with a comma delineator used in Hindu-Arabic numeration. Thus, 2 lines imply 2 comma delineations after the value, i.e. millions.

For example, \( \text{CDLXXV} = 475 \times 1000 = 475,000 \)
and \( \text{DCCXCVIICCCLXIXC} = (797 \times 1000 \times 1,000) + (369 \times 1,000) + 98 = 797,369,098 \)

**Chinese Abacus**

The Chinese Abacus is an ancient calculation device still used in certain parts of the world. Please note that this discussion will only cover representing numerical values on the abacus and not using the abacus for calculation. As would seem reasonable, the organization of the Chinese Abacus parallels that of the Chinese Rod system. The abacus frame is organized into vertical bars, each representing successive place values in the base ten system. The frame is vertically divided into two regions. The bottom region (the earth region) has 5 beads per bar and each bead is assigned a value of one. The top region (the heaven region) has two beads per bar and each bead is assigned a value of five. Beads are only “counted” when they are in contact with the part of the frame dividing the bars into the top and bottom regions or when they are in contact with beads already touching the part of the frame dividing the bars into the top and bottom regions. Unused beads are left at the top of the heaven region and at the bottom of the earth region. Translating to and from the Chinese Abacus is accomplished by writing the appropriate value for each bar. For example,
Using Multilink Cubes or other Manipulatives

(You can teach old numerals with new tricks!)

Realistically, it is most likely that you can’t find Egyptian, Mayan, or other exotic themed numeration manipulatives. For this reason, I recommend utilizing commonly available manipulatives, like multilink cubes, to represent the exotic symbols. Using the Mayan numeration system as an example, one could declare that white cubes represent clam shells (0’s), black cubes represent the dots (1’s), and yellow cubes represent the rods (5’s). With these simple designations, an effective visual and kinesthetic model is created. Quite astonishingly, this simple model is extremely effective in assisting students to make more meaningful connections to numeration and, thusly, promotes superior learning. Furthermore, by similarly using the same manipulatives to represent other numeration systems, comparison and contrast of varied numeration systems is facilitated as issues of specific symbolic representation are minimized while algorithmic and organizational rules are emphasized.

The Numeration Challenge

(Can we build it? Yes we can!)

To test your understanding and creativity with numeration, the following exercises have been created.

1.) Start with a set a 100 multilink cubes arranged into 10 colors with 10 cubes per color. Using these cubes, design a numeration system that allows you to represent every whole number from 0 to 100. You may use all 100 cubes, but it not necessary to do so.

2.) Now reduce your set to 50 cubes and use this set to design a numeration system that allows you to represent every whole number from 0 to 100. Again, it is not necessary to use all of the cubes in your set. The most important issue is the ability of your system to represent every value from 0 to 100.

3.) Now reduce you set to 25 cubes. Can you still design a numeration system that allows you to represent every whole number from 0 to 100? If so, describe the numeration system.

4.) Finally, design a numeration system that allows you to represent every whole number from 0 to 100 using the minimal number of cubes. How many cubes did you need?
Materials and Pricing Information

The only materials required to implement this activity into a K-12 or college classroom are a set of unifix or multilink cubes. Pricing of unifix or multilink cubes can obtained by contacting a myriad of sources including local sources like CC Educational Supply or internet sources like EAI Education (http://www.eaieducation.com/) or ETA Cuisenaire (http://www.etacuisenaire.com/). Costs will vary depending on class size and preference for individual or group work, but will probably fall in the range from $40 to $160.

Possible TEKS Correlations for Presentation

- 2nd Grade: 2.1A, 2.12A
- 3rd Grade: 3.1B, 3.1C,
- 4th Grade: 4.1B, 4.6B, 4.14A, 4.16
- 5th Grade: 5.14C, 5.14D, 5.16
- 6th Grade: 6.11
- 7th Grade: 7.13, 7.14
- 8th Grade: 8.14, 8.16