13.3 Arc Length and the Unit Tangent Vector

If \( \vec{v} = \langle v_1, v_2, v_3 \rangle \), then the length of \( \vec{v} \) is given by

\[
|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}
\]

If \( \vec{v} \) is a non-zero vector, then the vector \( \frac{1}{|\vec{v}|} \vec{v} \) is a vector of length one (unit vector) with the same direction as \( \vec{v} \).

1. Find a unit vector having the same direction as the vector from the point \( A(-1,0,2) \) to the point \( B(3,1,1) \).

Solution:

\[
\vec{AB} = \langle 3 - (-1), 1 - 0, 1 - 2 \rangle = \langle 4, 1, -1 \rangle
\]

\[
|\vec{AB}| = \sqrt{4^2 + 1^2 + (-1)^2} = \sqrt{18} = 3\sqrt{2}
\]

The unit vector with the same direction as \( \vec{AB} \) is

\[
\frac{1}{3\sqrt{2}} \langle 4, 1, -1 \rangle = \left\langle \frac{4}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, -\frac{1}{3\sqrt{2}} \right\rangle
\]

**Definition:** The unit tangent vector of a smooth curve \( \vec{r}(t) \) is

\[
\vec{T} = \frac{\vec{v}}{|\vec{v}|} \quad \text{where} \quad \vec{v} = \frac{d\vec{r}}{dt}
\]

**Definition:** The length of a smooth curve \( \vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}, a \leq t \leq b \), that is traced exactly once as \( t \) increases from \( t = a \) to \( t = b \), is

\[
L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt
\]

or

\[
\int_a^b |\vec{v}| \, dt
\]

Examples: Find the curve's unit tangent vector. Also, find the length of the indicated portion of the curve.

2. \( \vec{r}(t) = (6 \sin 2t)\hat{i} + (6 \cos 2t)\hat{j} + 5t\hat{k}, \quad 0 \leq t \leq \pi \)

Solution:
\[ \vec{r}'(t) = (12 \cos 2t) \hat{i} + (-12 \sin 2t) \hat{j} + 5 \hat{k} \]

\[ |\vec{r}'(t)| = \sqrt{(12 \cos 2t)^2 + (-12 \sin 2t)^2 + 5^2} \]

\[ = \sqrt{144 \cos^2 2t + 144 \sin^2 2t + 25} \]

\[ = \sqrt{144(\cos^2 2t + \sin^2 2t) + 25} \]

\[ = \sqrt{144 + 25} = \sqrt{169} = 13 \]

\[ \vec{t} = \frac{\vec{v}}{|\vec{v}|} = \left\langle \frac{12 \cos 2t}{13}, -\frac{12 \sin 2t}{13}, \frac{5}{13} \right\rangle \]

\[ L = \int_0^b |\vec{v}| \, dt = \int_0^\pi 13 \, dt = 13\pi \]

3. \( \vec{r}(t) = 6t^3 \hat{i} - 2t^3 \hat{j} - 3t^3 \hat{k}, \quad 1 \leq t \leq 2 \)

\[ \vec{v} = \vec{r}'(t) = 18t^2 \hat{i} - 6t^2 \hat{j} - 9t^2 \hat{k} \]

\[ |\vec{v}| = |\vec{r}'(t)| = \sqrt{(18t^2)^2 + (-6t^2)^2 + (-9t^2)^2} \]

\[ = \sqrt{324t^4 + 36t^4 + 81t^4} = \sqrt{441t^4} = 21t^2 \]

\[ \vec{t} = \frac{\vec{v}}{|\vec{v}|} = \left\langle \frac{18t^2}{21t^2}, \frac{-6t^2}{21t^2}, \frac{-9t^2}{21t^2} \right\rangle = \left\langle \frac{6}{7}, \frac{-2}{7}, \frac{-3}{7} \right\rangle \]

\[ L = \int_a^b |\vec{v}| \, dt = \int_1^2 21t^2 \, dt = 7t^3 \bigg|_1^2 = 7[8 - 1] = 49 \]

**Definition:** The arc length parameter of a smooth curve \( \vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k} \) with Base Point \( P(t_0) \) is

\[ s(t) = \int_{t_0}^t \sqrt{(x'(\tau))^2 + (y'(\tau))^2 + (z'(\tau))^2} \, d\tau = \int_{t_0}^t |\vec{v}(\tau)| \, d\tau \]

4. Find the arc length parameter along the curve \( \vec{r}(t) = (\cos t + t \sin t) \hat{i} + (\sin t - t \cos t) \hat{j} \) from the base point \( t = 0 \). Then find the length of the portion of the curve on the interval \( \frac{\pi}{2} \leq t \leq \pi \).

\[ \vec{r}'(t) = (-\sin t + \sin t + t \cos t) \hat{i} + (\cos t - \cos t + t \sin t) \hat{j} \]

\[ = (t \cos t) \hat{i} + (t \sin t) \hat{j} \]

\[ |\vec{r}'(\tau)| = \sqrt{(\tau \cos \tau)^2 + (\tau \sin \tau)^2} = \sqrt{\tau^2 \cos^2 \tau + \tau^2 \sin^2 \tau} = \sqrt{\tau^2 (\cos^2 \tau + \sin^2 \tau)} = \tau \]
\[ s(t) = \int_0^t |\ddot{v}(\tau)| d\tau = \int_0^t \dot{\tau} \, d\tau = \frac{1}{2} \tau^2 \bigg|_0^t = \frac{1}{2} t^2 \]

The length on the interval \(0 \leq t \leq \pi\) is \(s(\pi) = \frac{1}{2} \pi^2\).

The length on the interval \(0 \leq t \leq \frac{\pi}{2}\) is \(s\left(\frac{\pi}{2}\right) = \frac{1}{2} \left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{8}\).

Thus the length on the interval \(\frac{\pi}{2} \leq t \leq \pi\) is \(\frac{\pi^2}{2} - \frac{\pi^2}{8} = \frac{3\pi^2}{8}\).