7.3 The Law of Cosines

As mentioned in section 7.1, if we are given two sides and the included angle or three sides of a triangle, then a unique triangle is determined. These are the SAS and SSS cases and require using the law of cosines to solve.

**LAW OF COSINES**

In any triangle ABC, with sides of lengths a, b, and c,

- \( a^2 = b^2 + c^2 - 2bc \cos A \)
- \( b^2 = a^2 + c^2 - 2ac \cos B \)
- \( c^2 = a^2 + b^2 - 2ab \cos C \)

Guidelines for solving a triangle given SAS:

1. Find the third side using the *Law of Cosines*
2. Find the smaller of the two remaining angles using the *Law of Sines*
3. Find the remaining angle using \( \alpha + \beta + \gamma = 180^\circ \)

Guidelines for solving a triangle given SSS:

1. Find the largest angle using the *Law of Cosines*
2. Find either of the two remaining angles using the *Law of Sines*
3. Find the remaining angle using \( \alpha + \beta + \gamma = 180^\circ \)

Solve each triangle

1. \( A = 41.4^\circ, b = 2.78, c = 3.92 \)

   ![Triangle](triangle.png)

   **Solution**: Given SAS

   \[
   a^2 = b^2 + c^2 - 2bc \cos A \\
   a^2 = 2.78^2 + 3.92^2 - 2(2.78)(3.92) \cos 41.4^\circ \\
   a = \sqrt{2.78^2 + 3.92^2 - 2(2.78)(3.92) \cos 41.4^\circ} \\
   a = 2.60
   \]

   Angle B is the smaller of the two remaining angles

   \[
   \frac{\sin B}{b} = \frac{\sin A}{a} \Rightarrow \frac{\sin B}{2.78} = \frac{\sin 41.4^\circ}{2.60} \\
   \sin B = \frac{2.78 \sin 41.4^\circ}{2.60} \Rightarrow \text{So } B = \sin^{-1} \left( \frac{2.78 \sin 41.4^\circ}{2.60} \right) = 45^\circ
   \]

   \( 41.4^\circ + 45^\circ + C = 180^\circ \) so \( C = 93.6^\circ \)

   Solution \( A = 41.4^\circ, B = 45^\circ, C = 93.6^\circ, a = 2.60, b = 2.78, c = 3.92 \)
2. $B = 112.8^\circ$, $a = 6.28$, $c = 12.2$

Solution: Given SAS

\[ b^2 = a^2 + c^2 - 2ac \cos B \]
\[ b^2 = 6.28^2 + 12.2^2 - 2(6.28)(12.2) \cos 112.8^\circ \]
\[ b = \sqrt{6.28^2 + 12.2^2 - 2(6.28)(12.2) \cos 112.8^\circ} \]
\[ b = 15.7 \]

The smallest of the two remaining angles is $A$

\[ \sin \frac{A}{a} = \frac{\sin B}{b} \quad \text{so} \quad \sin \frac{A}{6.28} = \frac{\sin 112.8^\circ}{15.7} \]
\[ \sin A = \frac{6.28 \sin 112.8^\circ}{15.7} \quad \text{So} \quad A = \sin^{-1}\left(\frac{6.28 \sin 112.8^\circ}{15.7}\right) = 21.6^\circ \]

$112.8^\circ + 21.6^\circ + C = 180^\circ$, therefore $C = 45.6^\circ$

Solution: $A = 21.6^\circ$, $B = 112.8^\circ$, $C = 45.6^\circ$ $a = 6.28$, $b = 15.7$, $c = 12.2$

3. $a = 28$, $b = 47$, $c = 58$

Solution: Given SSS

The largest angle is the angle opposite the longest side which is angle $C$

\[ c^2 = a^2 + b^2 - 2ac \cos C \]
\[ 58^2 = 28^2 + 47^2 - 2(28)(47) \cos C \]
\[ 58^2 - 28^2 - 47^2 = -2(28)(47) \cos C \]
\[ \cos C = \frac{58^2 - 28^2 - 47^2}{-2(28)(47)} \quad \text{So} \quad C = \cos^{-1}\left(\frac{58^2 - 28^2 - 47^2}{-2(28)(47)}\right) = 98.1^\circ \]

\[ \frac{\sin A}{a} = \frac{\sin C}{c} \quad \text{So} \quad \frac{\sin A}{28} = \frac{\sin 98.1^\circ}{58} \]
\[ A = \sin^{-1}\left(\frac{28 \sin 98.1^\circ}{58}\right) = 28.6^\circ \]

$A + B + C = 180^\circ$

$28.6^\circ + 98.1^\circ + B = 180^\circ$, therefore $B = 53.3^\circ$

Solution: $a = 28$, $b = 47$, $c = 58$, $A = 28.6^\circ$, $B = 53.3^\circ$, $C = 98.1^\circ$
4. Airports A and B are 450 km apart, on an east-west line. Tom flies in a northeast direction from A to C. From C he flies 359 km on a bearing of 128.7° to B. How far is C from A?

\[ 450 \]

By corresponding angles, angle B is the complement of 51.3°, so angle B is 38.7°. By the law of cosines

\[ b^2 = a^2 + c^2 - 2ac \cos B \]

\[ b^2 = 359^2 + 450^2 - 2(359)(450) \cos 38.7° \]

\[ b = \sqrt{359^2 + 450^2 - 2(359)(450) \cos 38.7°} = 281 \text{ km} \]

5. A ship is sailing east. At one point, the bearing of a submerged rock is 45°20’. After the ship has sailed 15.2 miles, the bearing of the rock has become 308°40’. Find the distance of the ship from the rock at the latter point.

Solution: 308°40’ − 270° = 38°40’ and 90° − 45°20’ = 44°40’, therefore

\[ 180° - 38°40’ - 44°40’ = 96°40’ \]

\[ \frac{a}{\sin 44°40’} = \frac{15.2}{\sin 96°40’} \]

\[ a = \frac{(\sin 44°40’)15.2}{\sin 96°40’} = 10.8 \text{ miles} \]
6. An airplane flies 180 miles from point X at a bearing of 125°, and then turns and flies at a bearing of 230° for 100 miles. How far is the plane from point X?

The included angle is $180° - 55° - 50° = 75°$. By the law of cosines

$$b^2 = 180^2 + 100^2 - 2(180)(100)\cos 75°$$

$$b = \sqrt{180^2 + 100^2 - 2(180)(100)\cos 75°} = 182 \text{ miles}$$ from point X