7.2 The Law of Sines (Ambiguous Case)

If two angles and any side is given, then the given information will lead to exactly one triangle. However, if two sides and an angle opposite one of them is given, then a unique triangle is not always determined. When given (SSA) there are three possible outcomes:

- No triangle exists
- Exactly one triangle exists
- Two triangles exist

To determine if the given information leads to two triangles; add the supplementary angle of the angle found using the law of sines to the given angle. If this sum is less than $180^\circ$, then two triangles exist, otherwise there is only one triangle.

Solve the following triangles.

1. $A = 67^\circ$, $a = 100$, and $c = 125$

Solution:

\[
\frac{\sin A}{a} = \frac{\sin C}{c} \\
\frac{\sin 67^\circ}{100} = \frac{\sin C}{125} \\
\sin C = \frac{125 \sin 67^\circ}{100} = 1.1506
\]

Since $1.1506 > 1$, there is no triangle with $A = 67^\circ$, $a = 100$, and $c = 125$

2. $a = 50$, $b = 26$, $A = 95^\circ$

Solution:

\[
\frac{\sin A}{a} = \frac{\sin B}{b} \\
\frac{\sin 95^\circ}{50} = \frac{\sin B}{26} \\
\sin B = \frac{26 \sin 95^\circ}{50} = 0.51802 \\
B = \sin^{-1} 0.51802 = 31^\circ
\]

Check for two triangles by finding the supplement to this angle and adding the result to the given angle. The supplement is $149^\circ$ and since $149^0 + 76^0 = 216^0 > 180^0$, there exists only one triangle.

$A + B + C = 180^\circ$

$67^\circ + 31^\circ + C = 180^\circ$

$C = 82^\circ$

\[
\frac{a}{\sin A} = \frac{c}{\sin C} \\
\frac{100}{\sin 67^\circ} = \frac{c}{\sin 82^\circ} \\
c = \frac{100 \sin 67^\circ}{\sin 82^\circ} = 93
\]

Solution: $a = 50$, $b = 26$, $c = 93$, $A = 95^\circ$, $B = 31^\circ$, $C = 82^\circ$
3. $a = 12.4, b = 8.7, \text{ and } B = 36.7^\circ$

Solution: \[
\frac{\sin B}{b} = \frac{\sin A}{a}
\]
\[
\frac{\sin 36.7^\circ}{8.7} = \frac{\sin A}{12.4}
\]
\[
\sin A = \frac{12.4 \sin 36.7^\circ}{8.7} = 0.85179
\]
\[
A = \sin^{-1}(0.85179) = 58.4^\circ
\]

Check for the existence of two triangles. The supplementary angle is $121.6^\circ$ and the sum of this angle and the given angle is $121.6^\circ + 36.7^\circ = 158.3^\circ < 180^\circ$, therefore two triangles exist.

Solve the first triangle.

\[
A + B + C = 180^\circ
\]
\[
58.4^\circ + 36.7^\circ + C = 180^\circ
\]
\[
C = 84.9^\circ
\]

\[
\frac{c}{\sin C} = \frac{b}{\sin B}
\]
\[
\frac{c}{\sin 84.9^\circ} = \frac{8.7}{\sin 36.7^\circ}
\]
\[
c = \frac{8.7 \sin 84.9^\circ}{\sin 36.7^\circ} = 14.5
\]

One triangle is $a = 12.4, b = 8.7, c = 14.5, A = 58.4^\circ, B = 36.7^\circ, C = 84.9^\circ$

Solve for the 2nd triangle:

\[
A + B + C = 180^\circ
\]
\[
121.6^\circ + 36.7^\circ + C = 180^\circ
\]
\[
C = 21.7^\circ
\]

\[
\frac{c}{\sin C} = \frac{b}{\sin B}
\]
\[
\frac{c}{\sin 21.7^\circ} = \frac{8.7}{\sin 36.7^\circ}
\]
\[
c = \frac{8.7 \sin 21.7^\circ}{\sin 36.7^\circ} = 5.4
\]

2nd triangle is $a = 12.4, b = 8.7, c = 5.4, A = 121.6^\circ, B = 36.7^\circ, C = 21.7^\circ$
4. \(a = 9.72, b = 11.8,\) and \(B = 38.6^\circ\)

Solution:
\[
\frac{\sin B}{b} = \frac{\sin A}{a}
\]
\[
\frac{\sin 38.6^\circ}{11.8} = \frac{\sin A}{9.72}
\]
\[
\sin A = \frac{11.8 \cdot \sin 38.6^\circ}{9.72} = 0.51391
\]
\[
A = \sin^{-1}(0.51391) = 30.9^\circ
\]

Check for the existence of two triangles. The supplementary angle is \(149.1^\circ\) and the sum of this angle and the given angle is \(149.1^\circ + 38.6^\circ = 187.6^\circ > 180^\circ\), therefore only one triangle exists.

Solve the triangle.
\[
A + B + C = 180^\circ
\]
\[
30.9^\circ + 38.6^\circ + C = 180^\circ
\]
\[
C = 110.5^\circ
\]

\[
\frac{c}{\sin C} = \frac{b}{\sin B}
\]
\[
\frac{c}{\sin 110.5^\circ} = \frac{11.8}{\sin 38.6^\circ}
\]
\[
c = \frac{11.8 \cdot \sin 110.5^\circ}{\sin 38.6^\circ} = 17.7
\]

The triangle is \(a = 9.72, b = 11.8, c = 17.7, A = 30.9^\circ, B = 38.6^\circ, C = 110.5^\circ\)

5. \(B = 39.68^\circ, a = 29.81, b = 23.67\)

Solution:
\[
\frac{\sin B}{b} = \frac{\sin A}{a}
\]
\[
\frac{\sin 39.68^\circ}{23.67} = \frac{\sin A}{29.81}
\]
\[
\sin A = \frac{29.81 \cdot \sin 39.68^\circ}{23.67} = 0.804126
\]
\[
A = \sin^{-1}(0.804126) = 53.53^\circ
\]

Check for the existence of two triangles. The supplementary angle is \(126.47^\circ\) and the sum of this angle and the given angle is \(126.47^\circ + 39.68^\circ = 166.15^\circ < 180^\circ\), therefore two triangles exist.

Solve the first triangle.
\[
A + B + C = 180^\circ
\]
\[
53.53^\circ + 39.68^\circ + C = 180^\circ
\]
\[
58.4^\circ + 36.7^\circ + C = 180^\circ
\]
\[
C = 86.79^\circ
\]

\[
\frac{c}{\sin C} = \frac{b}{\sin B}
\]
\[
\frac{c}{\sin 86.79^\circ} = \frac{23.67}{\sin 39.68^\circ}
\]
\[
c = \frac{23.67 \cdot \sin 86.79^\circ}{\sin 39.68^\circ} = 37.01
\]

One triangle is \(a = 29.81, b = 23.67, c = 37.01, A = 53.53^\circ, B = 39.68^\circ, C = 86.79^\circ\)
Solve for the 2nd triangle:

\[
A + B + C = 180^\circ \\
126.47^\circ + 39.68^\circ + C = 180^\circ \\
C = 13.85^\circ
\]

\[
\frac{c}{\sin C} = \frac{b}{\sin B} \\
c = \frac{23.67}{\sin 13.85^\circ} = \frac{23.67 \sin 13.85^\circ}{\sin 39.68^\circ} = 8.87
\]

2nd triangle is \( a = 29.81, b = 23.67, c = 8.87, A = 126.47^\circ, B = 39.68^\circ, C = 13.85^\circ \)