5.4 Present Value of an Annuity and Amortization

The present value of an annuity is the amount that would have to be deposited in one lump sum today in order to produce exactly the same balance at the annuity’s maturity. Recall the future value of an ordinary annuity is

\[ S = R \left[ \frac{(1+i)^n - 1}{i} \right] \]

and if \( P \) dollars are deposited today at the same compound interest rate \( i \), then at the end of \( n \) payments, the amount in the account is \( P(1+i)^n \). This amount must be the same as the future value of the annuity; that is,

\[
P(1+i)^n = R \left[ \frac{(1+i)^n - 1}{i} \right]
\]

Find the following:

1. \( a_{18/0.04} \)

Solution: \( a_{18/0.04} = \frac{1-(1+0.04)^{-18}}{0.04} = 12.65930 \)

2. \( a_{32/0.027} \)

Solution: \( a_{32/0.027} = \frac{1-(1+0.027)^{-32}}{0.027} = 21.24704 \)

Find the present value of each ordinary annuity.

3. Payments of $15,806 quarterly for 3 years at 6.8% compounded quarterly.

Solution: The interest rate per period is \( i = \frac{0.068}{4} \), thus

\[
P = R \left[ \frac{1 - (1 + i)^{-n}}{i} \right] = 15,806 \left[ \frac{1 - \left(1 + \frac{0.068}{4}\right)^{-12}}{\frac{0.068}{4}} \right] = 170,275.47
\]
4. Payments of $18,579 every 6 months for 8 years at 7.4% compounded semiannually.
Solution: The interest rate per period is \( i = \frac{0.074}{2} \), thus
\[
P = R \left( \frac{1 - (1 + i)^{-n}}{i} \right) = 18,579 \left[ \frac{1 - \left( \frac{1 + 0.074}{2} \right)^{-16}}{\frac{0.074}{2}} \right] = $221,358.80
\]

5. Find the lump sum deposited today that will yield the same total amount as payments of $10,000 at the end of each year for 15 years at 6% compounded annually.
Solution:
\[
P = R \left( \frac{1 - (1 + i)^{-n}}{i} \right) = 10,000 \left[ \frac{1 - (1 + 0.06)^{-15}}{0.06} \right] = $97,122.49
\]

A loan is amortized if both the principal and interest are paid by a sequence of equal periodic payments.
\[
P = R \left( \frac{1 - (1 + i)^{-n}}{i} \right)
\]
\[
R = \frac{p}{\frac{1 - (1 + i)^{-n}}{i}} = \frac{Pi}{1 - (1 + i)^{-n}}
\]

**Amortization Payments**
A loan of P dollars at interest rate i per period may be amortized in n equal periodic payments of R dollars made at the end of each period, where
\[
R = \frac{Pi}{1 - (1 + i)^{-n}}
\]

Find the payment necessary to amortize each of the following loans.
6. $5500; 9.5% compounded monthly; 24 monthly payments.
Solution: \( R = \frac{Pi}{1 - (1 + i)^{-n}} = \frac{5500\left(\frac{0.095}{12}\right)}{1 - (1 + \frac{0.095}{12})^{-24}} = $252.53
\]

7. A $153,762 house at 5.45% for 30 years
Solution: The periodic interest rate is \( i = \frac{0.0545}{12} \) and there will be a total of 30(12) = 360 payments:
\[
R = \frac{Pi}{1 - (1 + i)^{-n}} = \frac{153,762\left(\frac{0.0545}{12}\right)}{1 - (1 + \frac{0.0545}{12})^{-360}} = $868.23
\]
8. Student borrowers now have more options to choose from when selecting repayment plans. The standard plan repays the loan in 10 years with equal monthly payments. The extended plan allows from 12 to 30 years to repay the loan. A student borrows $35,000 at 7.43% compounded monthly:

a) Find the monthly payment and total interest paid under the standard plan.
b) Find the monthly payment and total interest paid under the extended plan for 20 years.

Solution:

a) \[ R = \frac{P \cdot i}{1 - (1 + i)^{-n}} = \frac{35,000 \cdot (0.0743/12)}{1 - \left(1 + \frac{0.0743}{12}\right)^{-120}} = 414.18 \]

The total amount paid over 10 years is \(414.18 \times 120 = 49,701.60\) The total interest paid is \(49,701.60 - 35,000 = 14,701.60\)

b) \[ R = \frac{P \cdot i}{1 - (1 + i)^{-n}} = \frac{35,000 \cdot (0.0743/12)}{1 - \left(1 + \frac{0.0743}{12}\right)^{-240}} = 280.46 \]

The total amount paid over 10 years is \(280.46 \times 240 = 67,353.60\) The total interest paid is \(67,353.60 - 35,000 = 32,353.60\)

9. Jenni Ramirez plans to retire in 20 years. She will make 240 equal monthly contributions to her retirement account. One month after her last contribution, she will begin the first of 120 monthly withdrawals from the account. She expects to withdrawal $3500 per month. How large must her monthly contributions be in order to accomplish her goal if her account is assumed to earn interest of 10.5% compounded monthly for the duration of her contributions and the 120 months of withdrawals?

Solution:

She will need a total of \( P = R \left[ \frac{1 - (1 + i)^{-n}}{i} \right] = 3500 \left[ \frac{1 - (1 + \frac{1.05}{12})^{-240}}{\frac{1.05}{12}} \right] = 259,384.15 \)

In her retirement account the day she retires.

From 5.3, the future value of an ordinary annuity is \( S = R \left[ \frac{(1+i)^n-1}{i} \right] \), thus

\[ R \left[ \frac{(1 + \frac{1.05}{12})^{240} - 1}{\frac{1.05}{12}} \right] = 259,384.15 \]

\[ R = \frac{259,384.15}{\left[ \frac{(1 + \frac{1.05}{12})^{240} - 1}{\frac{1.05}{12}} \right]} = 320.03. \] She will need to contribute $320.03 per month for 20 years.
10. The Beyes plan to purchase a home for $212,000. They will pay 20% down and finance the remainder for 30 years at 7.2% interest compounded monthly.

a) How large are their monthly payments?

b) What will be the loan balance after they have made their 96th payment?

c) If they were to increase their monthly payments by $150, how long would it take to pay off the loan?

Solution

a) The will finance $212,000 - $212,000(0.20) = $169,600

Their payments will be

\[
R = \frac{P(1 - (1 + i)^{-n})}{i} = \frac{169,600 \left( \frac{0.072}{12} \right)}{1 - \left( 1 + \frac{0.072}{12} \right)^{-360}} = 1151.22 \text{ per month}
\]

b) \[y = R \left[ \frac{1 - (1 + i)^{-(n-x)}}{i} \right] = 1151.22 \left[ \frac{1 - \left( 1 + \frac{0.072}{12} \right)^{-96}}{\frac{0.072}{12}} \right] = 152,320.58\]

c) Increase the monthly payments by $150 will make the monthly payments $1302.22.

\[
P = R \left[ \frac{1 - (1 + i)^{-n}}{i} \right]
\]

\[
\frac{169,900}{1302.22} = \left[ \frac{1 - \left( 1 + \frac{0.072}{12} \right)^{-n}}{\frac{0.072}{12}} \right]
\]

\[
\left( \frac{169,900}{1302.22} \right) \cdot \left( \frac{0.072}{12} \right) = 1 - \left( 1 + \frac{0.072}{12} \right)^{-n}
\]

\[
\left( \frac{169,900}{1302.22} \right) - 1 = - \left( 1 + \frac{0.072}{12} \right)^{-n}
\]

\[
1 - \left( \frac{169,900}{1302.22} \right) \cdot \left( \frac{0.072}{12} \right) = \left( 1 + \frac{0.072}{12} \right)^{-n}
\]

\[
\ln \left( 1 - \left( \frac{169,900}{1302.22} \right) \cdot \left( \frac{0.072}{12} \right) \right) = \ln \left( 1 + \frac{0.072}{12} \right)^{-n}
\]

\[
\ln \left[ 1 - \left( \frac{169,900}{1302.22} \right) \cdot \left( \frac{0.072}{12} \right) \right] = - n \ln \left( 1 + \frac{0.072}{12} \right)
\]
\[
\frac{\ln\left[1 - \left(\frac{169,900}{1302.22} \cdot \frac{.072}{12}\right)\right]}{\ln\left(1 + \frac{.072}{12}\right)} = -n
\]

\[-255.26 = -n\]

Or \(n = 255.26\) months or between 21 and 22 years