5.3 Future Value of an Annuity and Sinking Funds

Definition: A sequence is a function whose domain is the set of positive integers. The values \( f(1), f(2), f(3), \ldots \) are called the terms of the sequence.
We usually write \( a_n \) instead of the function notation \( f(n) \), therefore the sequence \( f(1), f(2), f(3), \ldots \) is most often written as \( a_1, a_2, a_3, \ldots \)

1. Write the first five terms of the sequence defined by \( a_n = n^2 - 1 \)
Solution: \( a_1 = 1^2 - 1 = 0, \ a_2 = 2^2 - 1 = 3, \ a_3 = 3^2 - 1 = 8, \ a_4 = 4^2 - 1 = 15, \ a_5 = 5^2 - 1 = 24 \)
The first five terms are 0, 3, 8, 15, 24

Definition: A geometric sequence is a sequence of the form
\[ a, ar, ar^2, ar^3, ar^4, \ldots \]
The number \( a \) is the first term, and \( r \) is the common ratio of the sequence. The \( n^{\text{th}} \) term of a geometric is given by
\[ a_n = ar^{n-1} \]

Write the first five terms of each geometric sequence. What is the 15\(^{\text{th}} \) term?

2. \( a = 4, r = 3 \)
Solution:
\( a_1 = 4, a_2 = 4 \cdot 3 = 12, a_3 = 4 \cdot 3^2 = 36, a_4 = 4 \cdot 3^3 = 108, a_5 = 4 \cdot 3^4 = 324 \)
The first five terms are 4, 12, 36, 108, 324
The 15\(^{\text{th}} \) term is \( a_n = ar^{n-1}, a_{15} = 4 \cdot 3^{15-1} = 4 \cdot 3^{14} \)

3. \( a = 2, \ r = -\frac{1}{3} \)
Solution:
\( a_1 = 2, a_2 = 2 \cdot \left(-\frac{1}{3}\right) = -\frac{2}{3}, a_3 = 2 \cdot \left(-\frac{1}{3}\right)^2 = \frac{2}{9}, a_4 = 2 \cdot \left(-\frac{1}{3}\right)^3 = -\frac{2}{27}, \ a_5 = 2 \cdot \left(-\frac{1}{3}\right)^4 = \frac{2}{81} \)
The first five terms are 2, \(-\frac{2}{3}, \frac{2}{9}, -\frac{2}{27}, \frac{2}{81} \)
The 15\(^{\text{th}} \) term is \( a_{15} = 2 \cdot \left(-\frac{1}{3}\right)^{15-1} = \frac{2}{3^{14}} \)

4. Find the 4\(^{\text{th}} \) term of the geometric sequence \( a = 10,000, \ r = 1.01 \).
\( a_n = ar^{n-1} \)
\( a_4 = 10,000 \cdot 1.01^3 = 10,303.01 \)
The sum $S_n$ of the first $n$ terms of a geometric series can be written as

$$S_n = a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1}$$

To come up with a formula for $S_n$, multiply both sides of the equation by $r$ obtaining

$$rS_n = ra + ar^2 + ar^3 + ar^4 + \cdots + ar^n$$

Now subtract

$$rS_n - S_n = ar^n - a$$

$$S_n(r - 1) = a(r^n - 1)$$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

### Sum of Terms

If a geometric sequence has first term $a$ and common ratio $r$, then the sum of the first $n$ terms is given by

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

Find the sum of the first five terms of the following geometric sequences.

5. $a = 4$, $r = 0.3$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_5 = \frac{4(0.3^5 - 1)}{0.3 - 1} = 5.7004$$

6. $a = 100$, $r = 1.05$

Solution:

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_5 = \frac{100(1.05^5 - 1)}{1.05 - 1} = 552.563125$$

A sequence of equal payments made at equal periods of time is called an **annuity**. If the payments are made at the end of the periods, and if the frequency of payments is the same as the frequency of compounding, the annuity is called an **ordinary annuity**.
7. $5000 is deposited at the end of each year for the next 6-years in a money market account paying 4.5% interest compounded annually. Find the future value of this annuity.

Solution: Consider each payment as a separate investment, the future value of the annuity will be the sum of all six investments.

1st payment: \[ A = P \left(1 + \frac{r}{n}\right)^{nt} = 5000 \left(1 + \frac{0.045}{1}\right)^5 = 5000(1.045)^5 \]
2nd payment: \[ A = 5000(1.045)^4 \]
3rd payment: \[ A = 5000(1.045)^3 \]
4th payment: \[ A = 5000(1.045)^2 \]
5th payment: \[ A = 5000(1.045)^1 \]
6th payment: \[ A = 5000 \]

The future value of the annuity is thus: \[ 5000 + 5000(1.045)^1 + 5000(1.045)^2 + 5000(1.045)^3 + 5000(1.045)^4 + 5000(1.045)^5 \]

This is just the sum of a geometric sequence with \( a = 5000, r = 1.045, \text{ and } n = 6 \)

\[
\frac{a(r^n - 1)}{r - 1} = \frac{5000(1.045^6 - 1)}{1.045 - 1} = \$33,584.46
\]

<table>
<thead>
<tr>
<th>Future Value of an Ordinary Annuity</th>
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<tbody>
<tr>
<td>[ S = R \left(\frac{(1+i)^{n-1}}{i}\right) ] or [ S = R \cdot S_{n</td>
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Where

- \( S \) is the future value
- \( R \) is the payment at the end of each period
- \( i \) is the interest rate per period
- \( n \) is the number of compounding periods

7. Find \( S_{16|0.04} \)

Solution: \[ S_{n|i} = \frac{(1+i)^n-1}{i} \]
\[ S_{16|0.04} = \frac{(1+0.04)^{16}-1}{0.04} = 21.8245 \]

Find the future value for each of the following ordinary annuities.

8. \( R = \$20,000 \), 4.5% compounded annually for 12 years

\[ S = R \left(\frac{(1+i)^{n-1}}{i}\right) \]
\[ S = 20,000 \left(\frac{(1+0.045)^{12}-1}{0.045}\right) = \$309,280.64 \]
9. \( R = 20,000 \), 6% compounded quarterly for 12 years
\[
S = R \left( \frac{(1+i)^n - 1}{i} \right)
\]
\[
S = 20000 \left( \frac{(1+ \cdot 06/4)^{48} - 1}{.06/4} \right) = 1,391,304.39
\]

Find the periodic payment that will amount to the given sum under the given conditions if payments are made at the end of each period.

10. \$65,000; money earns 6% compounded semiannually for 4.5 years.
\[
S = R \left( \frac{(1+i)^n - 1}{i} \right)
\]
\[
65,000 = R \left( \frac{(1+0.06/2)^{9} - 1}{.06/2} \right)
\]
\[
R = \frac{65,000}{\left( \frac{(1+0.06/2)^9 - 1}{.06/2} \right)} = 6398.20
\]

11. \$9,000; money earns 7% compounded monthly for 2.5 years.
\[
S = R \left( \frac{(1+i)^n - 1}{i} \right)
\]
\[
9000 = R \left( \frac{(1+ \cdot 07/12)^{30} - 1}{.07/12} \right)
\]
\[
R = \frac{9000}{\left( \frac{(1+0.07/12)^{30} - 1}{.07/12} \right)} = 275.39
\]

If the payments are made at the beginning of the periods, and if the frequency of payments is the same as the frequency of compounding, the annuity is called an **annuity due**.

\[
\text{Future Value of an annuity due} \\
S = R \left( \frac{(1+i)^{n+1} - 1}{i} \right) - R
\]
Find the future value of each annuity due.

12. Payments of $16,000 for 11 years at 4.7% compounded annually

\[
S = R \left( \frac{(1 + i)^{n+1} - 1}{i} \right) - R
\]

\[
S = 16,000 \left( \frac{(1 + .047)^{12} - 1}{.047} \right) - 16,000 = $234,295.32
\]

13. Payments of $1500 for 11 years at 7% compounded quarterly

\[
S = R \left( \frac{(1 + i)^{n+1} - 1}{i} \right) - R
\]

\[
S = 1500 \left( \frac{(1 + .07/4)^{44+1} - 1}{.07/4} \right) - 1500 = $99,897.88
\]

14. Hassi is paid on the first day of the month, and $80 is automatically deducted from his pay and deposited in a savings account. If the account pays 7.5% interest compounded monthly, how much will be in the account after 3 years and 9 months.

\[
S = R \left( \frac{(1 + i)^{n+1} - 1}{i} \right) - R
\]

\[
S = 80 \left( \frac{(1 + .075/12)^{45+1} - 1}{.075/12} \right) - 80 = $4168.30
\]

15. Chuck Hickman deposits $10,000 at the beginning of each year for 12 years in an account paying 5% compounded annually. He then puts the total amount on deposit in another account paying 6% compounded semiannually for another 9 years. Find the final amount on deposit after the entire 21-year period.

Solution:

\[
S = R \left( \frac{(1 + i)^{n+1} - 1}{i} \right) - R
\]

\[
S = 10,000 \left( \frac{(1 + .05)^{12+1} - 1}{.05} \right) - 10,000 = $167,129.83
\]

\[
A = P \left( 1 + \frac{r}{n} \right)^{nt}
\]

\[
A = 167,129.83 \left( 1 + \frac{.06}{2} \right)^{18} = $284,527.35
\]