APPLICATIONS AND MODELING WITH QUADRATIC EQUATIONS
Now that we are starting to feel comfortable with the factoring process, the question becomes “what do we use factoring to do?” There are a variety of classic applications that require the ability to solve a quadratic equation. A quadratic equation is an equation that contains an ‘x squared’ term along with possibly an ‘x’ term and a ‘constant’ term. Some examples would be: \( x^2 - 2x = 8, \ x^2 = 4x, \) and \( x^2 + 4x + 3 = 0. \) The way we handle all such equations is to first set the equation equal to zero by adding or subtracting the same terms to each side of the equation, and then look for a factorization. The reason we do this is simple, let’s look at an example to help understand.

Solve \( x^2 - 2x = 8. \) The initial problem here is that there are many (infinitely many) ways that two terms can subtract and give us ‘8’. So, we will subtract the 8 from each side of the equation to get:

\[
x^2 - 2x - 8 = 0
\]

We will then factor to get:

\[
(x - 4)(x + 2) = 0
\]

This equation is actually giving us all of the information we need to solve the equation. Where there was initially a problem because there are many different ways to get a subtraction equal to ‘8’, we now have a product (multiplication) equal to zero (we are multiplying \( x-4 \) times \( x+2 \)). But, the only way two things will multiply and give us a zero is if one of the things we are multiplying is zero (does this make good number sense?) This leads to the two equations:

\[
x - 4 = 0 \quad or \quad x + 2 = 0
\]

And then, the two solutions: \( x = 4 \) or \( x = -2 \) (\textit{do you see how?})

The purpose of this seminar is to look at how to set up word problems that will lead to quadratic equations, and also to look at some classic ‘models’ that require quadratic equations. Then, we will use what we have learned about factoring to solve these problems.
As with most ‘word problems’ or ‘story problems’, the initial challenge is to set up the equation that we need to solve. To do this, often requires a ‘translation’ from words to mathematical expressions.

Examples:

<table>
<thead>
<tr>
<th>Words</th>
<th>Mathematical Expressions</th>
</tr>
</thead>
<tbody>
<tr>
<td>‘Three less than two times a number’</td>
<td>$2x - 3$ for 'number' $x$</td>
</tr>
<tr>
<td>‘Three less than the square of a number’</td>
<td>$x^2 - 3$ for 'number' $x$</td>
</tr>
<tr>
<td>‘The sum of a number and it’s square’</td>
<td>$x^2 + x$ for 'number' $x$</td>
</tr>
<tr>
<td>‘Four more than a number’</td>
<td>$x + 4$ for 'number' $x$</td>
</tr>
<tr>
<td>‘Two consecutive integers’</td>
<td>$x$ and $x + 1$</td>
</tr>
<tr>
<td>‘Two consecutive even integers’</td>
<td>$x$ and $x + 2$</td>
</tr>
<tr>
<td>‘Two consecutive odd integers’</td>
<td>$x$ and $x + 2$</td>
</tr>
</tbody>
</table>

Without a correct ‘set up’ for a word problem, we will not be able to find the correct solution. This part of any word problem is crucial. Define a variable for a quantity you are looking for and then begin thinking about what you know about this quantity as it relates to the given information.

Some of the most common word problems for quadratic equations relate to area, perimeter, and numbers (including consecutive types of integers). All that we will need to attack these types of questions are some basic geometric formulas.

Squares

\[
\text{Area: } A = s^2 \\
\text{Perimeter: } P = 4s
\]

Rectangles

\[
\text{Area: } A = l \cdot w \\
\text{Perimeter: } P = 2l + 2w
\]
Before we get to working some problems together, here are two good examples to illustrate why ‘looking’ for the answer (trial and error) is not a good method to rely on for problem solving.

Example 1:

The length of a rectangle is one more than twice the width. The area of the rectangle is 6 square feet. Find the dimensions of the rectangle.

1. Draw a picture of a rectangle with a length and width. We know that the length is one more than twice the width, so \( l = 2w + 1 \).

2. We also know that the area is 6 square feet, which gives us the equation:

\[
(2w + 1) \cdot w = 6
\]

3. Solve the equation.

Distribute to get: \( 2w^2 + w = 6 \)
Subtract 6 from each side \(-6 - 6\)
To get: \( 2w^2 + w - 6 = 0 \)
Factor to get: \( (2w - 3)(w + 2) = 0 \)
Solve each factor equal to zero: \( 2w - 3 = 0 \) or \( w + 2 = 0 \)
This gives us the solutions: \( w = \frac{3}{2} = 1.5 \) or \( w = -2 \)

Since distance must be positive, the width can’t be ‘-2’.
So \( w = 1.5 \) feet and \( l = 2 \cdot 1.5 + 1 = 3 + 1 = 4 \) feet

The width is 1.5 feet and the length is 4 feet. (would you have ‘guessed’ 1.5 feet?)
Example 2:

Two consecutive even integers are such that the square of the smaller is four more than three times the larger. Find the integers.

1. Two consecutive even integers are: $x$ and $x + 2$

2. The larger of these is $x + 2$ and the smaller is $x$.

3. So, four more than three times the larger is: $3(x + 2) + 4$
   And the square of the smaller is: $x^2$.

4. The equation we need to solve is: $x^2 = 3(x + 2) + 4$

5. Solve the equation.

Distribute to get: $x^2 = 3x + 6 + 4$
Combine like terms to get: $x^2 = 3x + 10$
Subtract the 3x and the 10 from each side to get: $x^2 - 3x - 10 = 0$

Factor to get: $(x - 5)(x + 2) = 0$

Solve each factor equal to zero to get: $x = 5$ or $x = -2$

6. Since 5 is not an even integer, this can’t be part of the solution. So the solution is that $x = -2$ and $x + 2 = -2 + 2 = 0$

The consecutive even integers are -2 and 0.

What would the solution be if we were asking for consecutive odd integers?

Three important things to notice here: first, would you have ‘looked’ for a negative solution; second, notice that we needed to go calculate the ‘second part’ of our solution once we identified ‘-2’ as the correct ‘x’; and third, 0 is an even number.

Let’s try some together.
The key to ‘modeling’ questions is that you will be given a ‘model’. What this means is that you will be given an equations that describes some situation and will be asked to use the equation. The real challenge here is to be clear on what each variable in the equation stands for, and be clear on which variable you are given a value for and which variable you are supposed to find.

Example:

The number of mosquitoes in Corpus Christi on Labor Day is modeled by the equation:

\[ M = 10r - r^2 \]

Where \( M \) is millions of mosquitoes, and \( r \) is the rainfall in inches in July.

a) How much rain is needed to have 16 million mosquitoes in Corpus on Labor Day?

Here, we are given the number of mosquitoes \( M = 16 \) and must find \( r \).

Solve the equation \( 16 = 10r - r^2 \) for \( r \).

Subtract the 16 from each side to get: \( 0 = -16 + 10r - r^2 \)

Factoring is much easier with a positive square term, so multiply both sides by ‘-1’ to get:

\[ 0 = 16 - 10r + r^2 \]

Rearrange to get: \( 0 = r^2 - 10r + 16 \)

Factoring, we get: \( 0 = (r - 2)(r - 8) \)

This gives us the two equations: \( r - 2 = 0 \) or \( r - 8 = 0 \).

Solving these gives us: \( r = 2 \) or \( r = 8 \).

To have 16 million mosquitoes on Labor Day in Corpus, we need to get either 2 inches of rain in July, or get 8 inches of rain in July.
b) How many mosquitoes will there be on Labor Day in Corpus if there is 3 inches of rain in July?

Here, we are given the rainfall \( r = 3 \text{ inches} \) and must calculate \( M \).

\[
M = 10(3) - (3)^2 = 30 - 9 = 21
\]

Since \( M \) is in millions of mosquitoes, there will be 21 million mosquitoes on Labor Day in Corpus if we get 3 inches of rain in July.

c) How much rainfall is needed to have no mosquitoes in Corpus for Labor Day?

‘No mosquitoes’ would mean that \( M = 0 \), so we will solve the equation:

\[
0 = 10r - r^2
\]

We can factor this easily since the equation is already set equal to zero and there is an ‘\( r \)’ that is common to both terms on the right side of the equation.

\[
0 = r(10 - r)
\]

This gives us the two equations: \( r = 0 \) or \( 10 - r = 0 \).

Solving we get: \( r = 0 \) or \( r = 10 \).

If there is no rain in July, or 10 inches of rain in July, then we will have no mosquitoes for Labor Day in Corpus.