ac Method of Factoring Trinomials

How do we turn: 

$$3x^2 - x - 4$$

Into:

$$(3x - 4)(x + 1)$$

(And… why might this be important?)
Traditionally, we factor for two main reasons. First, to be able to solve equations; and second, to be able to simplify ‘rational expressions’. Next semester in Math 0373, you will start by working with the rational expressions. We will focus on the solving equations aspect in the last seminar for Math 0371. The problem with trying to solve an equation like $3x^2 + 5x = 2$ (or any other equation that contains both an ‘x squared’ term and an ‘x’ term is that many different combinations will add to equal ‘2’). What we do first is set the equation equal to zero. For $3x^2 + 5x = 2$, we subtract the ‘2’ from both sides of the equation and get: $3x^2 + 5x - 2 = 0$. Now, if we can factor our trinomial to get $(3x - 1)(x + 2) = 0$; then we can make use of the fact that things multiply and give us zero only if one of the things we are multiplying is zero (does this make good arithmetic sense?). If one of our factors must be zero, then we get the two equations: $3x - 1 = 0$ and $x + 2 = 0$. We can now solve both of these equations to find that $x = \frac{1}{3}$ or $x = -2$.

The purpose of this seminar is to continue the discussion on how to factor a general trinomial: $ax^2 + bx + c$. The last seminar was focused on a ‘trial and error’ approach. Many students are not as comfortable with this type of approach and feel better if they have a ‘step by step’ procedure. The ‘ac method’ is a step by step procedure for factoring trinomials.

As we begin, let’s first review some terminology for the general trinomial. $ax^2 + bx + c$

The ‘coefficients’ of the trinomial are $a, b, and c$. These are the ‘numbers’. The $ax^2$ is called the ‘square’ or ‘quadratic’ term, the $+bx$ is called the ‘linear’ or ‘mixed’ term, and the $+c$ is called the ‘constant’ term (because it has no variable, it’s value is ‘constant’). Remember that for the linear term and the constant term, you must take the sign that each has in the trinomial you are working with. For example; if your trinomial is $5x^2 - 7x + 10$, then $a = 5, b = -7, and c = +10$. As you hopefully remember from the last seminar on factoring, the real challenge is to get the $bx$ (linear term) when you multiply your factors together. This is connected to the product $a \cdot c$, hence the name ‘ac method’.
As the name suggests, this method of factoring starts by looking at the product: \( a \cdot c = N \). We must find two numbers \( A \) and \( B \), such that two things happen: \( A \cdot B = N \) and \( A + B = b \). Where \( N \) is our product \( a \cdot c \) and \( b \) is our linear coefficient (from \( bx \) in the trinomial). Let’s practice this game with \( N = 40 \).

We must first be clear on all the ways to make the ‘40’ by multiplying two numbers. Find all combinations \( A \) and \( B \) with \( A \cdot B = 40 \).

\[
1 \cdot 40 = 40, (-1) \cdot (-40) = 40, 2 \cdot 20 = 40, (-2) \cdot (-20) = 40,
4 \cdot 10 = 40, (-4) \cdot (-10) = 40, 5 \cdot 8 = 40, (-5) \cdot (-8) = 40
\]

Notice that there will always be two different ‘mixtures’ of sign that will give us the correct sign for \( N \). Now for the second part….consider the following \( b \)'s.

\[
\text{If } b = 41, \text{ then } A = 1 \text{ and } B = 40 \text{ because } 1 + 40 = 41
\]
\[
\text{If } b = -41, \text{ then } A = -1 \text{ and } B = -40 \text{ because } (-1) + (-40) = -41
\]
\[
\text{If } b = 22, \text{ then } A = 2 \text{ and } B = 20 \text{ because } 2 + 20 = 22
\]
\[
\text{If } b = -22, \text{ then } A = -2 \text{ and } B = -20 \text{ because } (-2) + (-20) = -22
\]
\[
\text{If } b = 14, \text{ then } A = 4 \text{ and } B = 10 \text{ because } 4 + 10 = 14
\]
\[
\text{If } b = -14, \text{ then } A = -4 \text{ and } B = -10 \text{ because } (-4) + (-10) = -14
\]
\[
\text{If } b = 13, \text{ then } A = 5 \text{ and } B = 8 \text{ because } 5 + 8 = 13
\]
\[
\text{If } b = -13, \text{ then } A = -5 \text{ and } B = -8 \text{ because } (-5) + (-8) = -13
\]

**NOTICE**: If \( N = 40 \), then the only linear terms that will be ‘factorable’ are:

\[
bx = \pm 41x, \text{ or } \pm 22x, \text{ or } \pm 14x, \text{ or } \pm 13x
\]

**If our linear term is not one of the above, then our trinomial does not factor!**

This little fact makes it possible to determine when we have a trinomial that does not factor. Anytime we can’t make ‘\( b \)’ with any possible combination of \( A \) and \( B \), then our trinomial does not factor!
Once we have found \( A \) and \( B \) with both \( A \cdot B = N \) and \( A + B = b \), we are ‘often’ only ‘half’ done. Why ‘often’? If we are lucky enough to have either \( a = 1 \) or \( c = 1 \), then the trinomial factors to:

\[
(x + A)(x + B) \text{ if } a = 1 \text{ or } (Ax + 1)(Bx + 1) \text{ if } c = 1
\]

However, ‘often’ we do not have either \( a \) or \( c \) equal to ‘1’. In this case, we can use factoring by grouping to finish factoring.

**FACTORING BY GROUPING APPROACH**

When we find \( A \) and \( B \), we have actually found how the linear term ‘\( bx \)’ was made when we multiplied the two factors we are looking for. This is to say that, when we multiply the factors we are looking for, we end up with \( Ax + Bx = bx \). This tells us that to factor our trinomial, we must ‘break up’ the linear term into these two parts and then use factoring by grouping on the pieces \( ax^2 + Ax \) and \( Bx + c \).

**Example 1: factor** \( x^2 - 7x + 6 \)

1. Since \( a = 1 \) and \( c = 6 \), \( a \cdot c = 1 \cdot 6 = 6 \). The only pairs that multiply to 6 are 1,6 or 2,3.

2. Since the 6 is positive, both factors must be positive, or both must be negative. This gives us four possibilities: 1 and 6, -1 and -6, 2 and 3, -2 and -3. These are the only pairs that will multiply to equal positive 6, and the only pair that will add to equal \( b = -7 \) is -1 and -6.

3. \[
x^2 - 7x + 6 = x^2 - 1x - 6x + 6 = x(x - 1) - 6(x - 1) = (x - 1)(x - 6)
\]

Now, the factor \( x - 1 \) is common to both pieces and can be factored out. This leaves us with the other factor of \( x - 6 \).

It was not a coincidence that here we found \( A = -1 \) and \( B = -6 \) and the polynomial factored as \( (x - 1)(x - 6) \). But again, this was only a by-product of the fact that \( a = 1 \) in the trinomial we were trying to factor, the general case when \( a \neq 1 \) requires more work and closer attention.
Example 2 (more challenging) factor $4x^2 + 5x - 6$.

1. Since $a = 4$ and $c = -6$, $a \cdot c = (4)(-6) = -24$. There are several pairs that multiply to 24:
   1,24 or 2,12 or 3,8 or 4,6.

2. Since -24 is negative, we need one positive and one negative factor. This leads to the following possibilities: -1 and 24, 1 and -24, -2 and 12, 2 and -12, -3 and 8, 3 and -8, -4 and 6, 4 and -6. These are the only pairs that will multiply to -24, and the only pair that will also add to equal 5 is: -3 and 8.

3. $4x^2 + 5x - 6 = 4x^2 - 3x + 8x - 6 = x(4x - 3) + 2(4x - 3)$
   
   Notice that the factor $(4x - 3)$ is now common to both pieces. Factoring this from both pieces leaves us the other factor of $(x + 2)$. We now have our factored trinomial: $4x^2 + 5x - 6 = (4x - 3)(x + 2)$.

As always, we should look for any common factors in the trinomial before starting this process. If there is a common factor, factor it from the trinomial and move forward with the process on the trinomial you are left with (the numbers we work with will be as small as possible this way).

Perhaps the most important thing to realize here is that factoring is a challenge. There is no ‘quick fix’ unless you are numerically gifted and just ‘see’ how things factor quickly. For those students who do not just ‘see’ how trinomials factor, the $ac$ method provides a nice step by step process for determining if a given trinomial factors and then finding the factorization. You may also be pleasantly surprised…..after finding many different factorizations, you may start to ‘see’ how trinomials factor without going through these steps. The knowledge about $A$ and $B$ can still be very useful. Remember, this tells you how the linear term $bx$ ‘came to be’ when the desired factors are multiplied. With this in mind, let’s look at one of the factorable trinomials from our worksheet.
Consider $4x^2 - 16x + 15$. We found that $N = 60$ with $A = -6$ and $B = -10$.

Now, when we multiply two factors $(Px + Q)(Rx + S)$ we get $a = P \cdot R$ and $b = Q \cdot S$ because the square term must come from multiplying the $x$ terms of each factor and the constant term must come from multiplying the constant term of each factor. We know that the linear term in our trinomial will come from $(-6x) + (-10x) = -16x$ and we see that the $Ax = P \cdot S$ and $Bx = Q \cdot Rx$. We can now go look at the factors of 4 and 15 (a and c) to determine how this happens.

<table>
<thead>
<tr>
<th>Pairs that multiply to ‘4’</th>
<th>Pairs that multiply to ‘15’</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 \cdot 4</td>
<td>1 \cdot 15</td>
</tr>
<tr>
<td>2 \cdot 2</td>
<td>3 \cdot 5</td>
</tr>
</tbody>
</table>

We are looking for one pair that multiplies to 4 and another pair that multiplies to 15 such that we can pair factors from each to give us the -6 and the -10. To this end, we can take $2 \cdot 2$ for 4 and $3 \cdot 5$ for 15. We can take $2 \cdot (-3) = -6$ and $2 \cdot (-5) = -10$. This tells us that:

$$4x^2 - 16x + 15 = (2x - 3)(2x - 5)$$

$$2x \cdot (-5) = -10x$$

$$(-3) \cdot 2x = -6x$$

The bottom line here is that factoring takes practice, no matter how you choose to do it. In the next seminar, we will discuss some special patterns that show up enough to justify looking at them separately so that when we see them, we can simply apply the ‘rule’ instead of going through either the ‘fill in the blank’ approach or the $ac$ method.