A Systematic Approach to Factoring

Step 1

Count the number of terms.

(\textit{Remember}****Knowing the number of terms will allow you to eliminate unnecessary tools.)

Step 2

Is there a \textbf{greatest common factor}? \textbf{Tool 1} (Page 3)

(\textit{Remember}****This is the most important tool. It is often used to hide other tools. When you factor the greatest common factor, it allows you to recognize other tools.)

Step 3

Can you \textbf{group} the expression? \textbf{(Grouping)} \textbf{Tool 2} (page 6)

(\textit{Remember}****Grouping requires 4 terms. If there are four terms present, the object is to group the first two and factor them. You will then have to factor the second two terms and try to make them look the first two terms.)

Step 4

Can you use the \textbf{“AC” method}? \textbf{Tool 3} (page 7)

(\textit{Remember}****The “A times C” method requires the problem to have three terms. Then number that we call “A” when multiplied by the number called “C” must have a sum that equals the middle term. We call the middle term “B”. We say that “A times C” must equal B.)

\[Ax^2 + Bx + C\]
Step 5

Can the problem be factored as the difference of two perfect squares. Tool 4

(Remember ****This tool requires two terms. Both the first and second terms must be perfect squares. A minus sign must separate the two terms. A plus “+” sign cannot separate the difference of two squares.)

Step 6

Is the problem the sum/difference of perfect cubes? Tool 5

(Remember ****Does the first and second terms have a cube root? This tool requires that a polynomial to have exactly two terms. The problem can either have the sum or the difference of two perfect cubes. Variables that are perfect cubes have exponents that are divisible by three.)
Polynomial Factoring

Factoring polynomials is one of the most important topics covered in Elementary Algebra. We will divide the concept into five tools.

**Tool 1**

**Factoring the greatest common monomial: (The most important tool!!!!)**

A monomial is a one termed polynomial. For example: \(3x, 5x^2r^5, 6, -3v\)

The term “factor” in this context means to divide. We are going to divide the largest factor found in each term of a polynomial from that polynomial. This is the most important of the five tools that we will discuss.

**Important Terms:**

- **Factor** – The original definition of the factor is an algebraic expression that is multiplied.

  Examples: 12 has factors 3 x 4 and 2 x 6
  \(3x^2y^4z\) has factors 3, \(x^2\), \(y^4\), \(z\)  When factors are variables, they are simply algebraic factors.

  We say in words that this expression is 3 times \(x^2\) times \(y^4\) times \(z\). The key is that these expressions (algebraic expressions) are separated by multiplication. ***Remember, factors are multiplied and factoring is dividing.***

- **Factoring** – The changing of an addition/subtraction problem into a multiplication problem.

- **Term** – A term is an algebraic expression separated by either a plus sign or a minus sign.

  Example: \(3x^2 + 5y^3z – 8m + 7\)

- **Constant** – A number that does not have a variable associated with it.

- **Combine** – To add or subtract. (Which ever is implied in the problem.)

- **Numerical Coefficient (coefficient)** – The number that is multiplied by the variable.
$3x^2 + 5y^3z - 8m + 7$

This algebraic expression has four terms. The first term $3x^2$ has two factors. This implies that a factor is not a terms but factors make up a term. The second term in the expression is $5y^3z$. This term has three factors. The factors are $5$, $y^3$, and $z$. These factors are separated by an understood multiplication symbol. ***Remember, factors are multiplied.***

**Factoring the greatest common monomial**

$3x^2yz^3 - 6x^3y^4z^2 + 12xy^2z^5$

The term “common” implies that each factors that you are looking for have something in common. You must first understand that there are three terms in this expression. $3x^2yz^3$ is the first term. $- 6x^3y^4z^2$ is the second term, and $12xy^2z^5$ is the third term. You are looking for the largest factor that is common to all three terms. In the end, you will divide each of these terms by this greatest common factor, (greatest common monomial) that you are trying to create.

**You are looking for common factors in each term.**

$3x^2yz^3m - 6x^3y^4z^2 + 12xy^2z^5m$  

The numerical coefficients $3$, $6$, and $12$ have a greatest common factor of $3$. This $3$ becomes a part of the greatest common factor.

$3x^2yz^3m - 6x^3y^4z^2 + 12xy^2z^5m$  

The variable $x^2$, $x^3$, and $x$ are common variables in that they are all $x$’s. The smallest variable ($x$) is the only one that can divide evenly into each term. This means that $x$ is part of the greatest common factor.

$3x^2yz^3m - 6x^3y^4z^2 + 12xy^2z^5m$  

The variables $y$, $y^4$, and $y^2$ are all common in that each term has at least one variable ($y$). The smallest number of $y$’s present is $y^1$ or simply $y$. This $y$ becomes part of the greatest common factor.

$3x^2yz^3m - 6x^3y^4z^2 + 12xy^2z^5m$  

The variables $z^3$, $z^2$, and $z^5$ are all common in that each term has at least one variable ($z$). The smallest number of $z$’s present is $z^2$. This variable $z^2$ becomes the final part of the greatest common factor.

The last factor in the first term is $m$. The variable $m$ cannot be a part of the greatest common factor because there is not at least one $m$ in each of the terms.

Elementary & Intermediate Algebra
The greatest common factor (GCF) is $3xyz^2$. You will divide each term by this GCF.

***Remember when you divide polynomials, you actually divide the numerical coefficients, but you simply subtract the exponents of the variables.*** The result is $xz^m$ from the first term, $2x^2y^3$ from the second term, and $4yz^3m$ from the third term. The answer should be written as follows: (*The signs stay consistent.*)

$$3xyz^2 \left( xz^m - 2x^2y^3 + 4yz^3m \right).$$

You should recognize that this problem has been factored because it is a multiplication problem. (**Remember,** Factoring is the changing of an addition/subtraction problem into a multiplication problem. This rule carries over into any type of polynomial. When you are looking for a GCF this is what should be done. Again, it must be said that this is the most important of the five tools that we are going to discuss. It can be a part of each of the remaining four tools.

**Simplify the following.**

$$5x^2(3m + 1) - 7x^5(3m + 1) + 4x^3(3m + 1)$$

$$5x^2(3m + 1) - 7x^5(3m + 1) + 4x^3(3m + 1)$$

This polynomial has three terms. The minus sign separates the first two terms and the plus sign separates the last two terms. The first term has three factors. The first factor is 5, the second is $x^2$, and the third factor is the expression $(3m + 1)$.

You should notice right away that the factors 5, 7, and 4, are all numerical coefficients, but they do not share a GCF. This implies that my overall GCF will not have a numerical coefficient. Each term has $\{x\}$ to some degree. The first term has $x^2$ which implies that there are **two** $x$’s. The second term has **five** $x$’s and the third term has **three** $x$’s. When we ask, “What is the largest number of $x$’s present in each term?” You can see right away that the number in question is not going to be the largest number of $x$’s present overall. The largest number of $x$’s present is $x^3$, but you can only divide each term by $x^2$ because you cannot subtract 5 from 2 and have a positive exponent. Obviously, the expression $(3m + 1)$ is common in all three terms. (**Remember,** the expression $(3m + 1)$ is a factor just the same as any other factor; you must treat it as you would any other factor.****

**Solution:**

$$x^2(3m + 1) \left( 5 - 7x^3 + 4x \right)$$
Tool 2

Grouping

**Grouping** – Grouping requires four or more terms. However, there must be an even number of terms. When using Tool 2, you must guarantee that you cannot use Tool 1 first.

**Procedure:**

\[6x^2 - 4x - 3xa + 2a\]

Group the first two terms. \((6x^2 - 4x)\) Then group the second two terms. \((- 3xa + 2a)\)

Find the GCF for the first two terms. Then find the GCF for the second two terms.

\[2x(3x - 2) - a(3x - 2)\]

In this case you must factor \(-a\). Only factoring \(-a\) will make the two expressions have something in common. Notice, factoring \(-a\) produces expressions which look exactly the same.

You are left with a two termed expression:

\[2x(3x - 2) - a(3x - 2)\]

You can now factor the expression \((3x - 2)\) from both terms.

\[(3x - 2)(2x - a)\]

That completes Tool 2.
Tool 3

The AC Method

Tool 3 is called the AC method. This is short for the A * C method. The AC method requires that a polynomial have exactly three terms. When we use Tool 3, we are simply taking a three termed polynomial and changing it into a four termed polynomial so we can group factor it by grouping (Tool 2).

We will introduce the quadratic equation in standard form:

$$Ax^2 + Bx + C$$

$$6x^2 - x - 1$$

You will notice that the number that represents \{A\} is 6 and the number that represents \{C\} is -1. So

\[ A = 6 \text{ and } C = -1 \]

When we say A * C, what we are really saying is 6 * -1. In this case A * C = -6.

Now we must find factors of \{-6\}:

Factors:

\[ 1 \times 6 \]

\[ 2 \times 3 \]

You must look for the factors of \{-6\} whose sum produces the coefficient of the middle term. The middle term’s coefficient is \{-1\}. The only way to get \{-1\} is by adding the factors in a very specific way. The factors \{1, 6\} cannot be combined (added/subtracted) to equal -1. However, the factors \{2, 3\} can be added to result in a -1 only when the combination -3 + 2 is present. So only when 3 is negative and 2 is positive will we obtain -1 as the difference. The rule says that the largest factor must have the same sign as the middle term. You must then replace the \(-x\) (the middle term) with its equivalent expression.

\[-x = -3x + 2x\]

\[6x^2 - x - 1\]

\[6x^2 - 3x + 2x - 1\]

You will notice that you have a nice three termed polynomial into a four termed polynomial. This changed was made so that we can now use the grouping method (Tool 2).

\[(6x^2 - 3x) + (2x - 1)\

\[3x(2x - 1) + 1(2x - 1)\]
(3x + 1) (2x – 1) \leftrightarrow \text{Factored!}

****Remember**** Grouping requires that the problem to have four terms and an \( \{ A * C \} \) has three terms. Logic says that if you want to group, you must make this three termed polynomial into a four termed polynomial. Replacing the middle term with two terms accomplishes this requirement.

Examples:

<table>
<thead>
<tr>
<th>Algebraic Expression</th>
<th>( A * C )</th>
<th>Replacement Factors for Middle Term</th>
<th>Results</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 12x^2 - 17x + 5 )</td>
<td>( 12 \times 5 )</td>
<td>(- 12x - 5x)</td>
<td>( 12x^2 - 12x - 5x + 5 )</td>
</tr>
<tr>
<td>( 6x^2 + 41xy + 7y^2 )</td>
<td>( 6 \times 7 )</td>
<td>( 42xy - xy )</td>
<td>( 6x^2 + 42xy - xy + 7y^2 )</td>
</tr>
<tr>
<td>( 2x^2 - 11x - 40 )</td>
<td>( 2 \times 40 )</td>
<td>(- 16x + 5x)</td>
<td>( 2x^2 - 16x + 5x - 40 )</td>
</tr>
<tr>
<td>( 4y^2 - 15y + 9 )</td>
<td>( 4 \times 9 )</td>
<td>(- 12y - 3y)</td>
<td>( 4y^2 - 12y - 3y + 9 )</td>
</tr>
</tbody>
</table>

Tool 4

The Difference of Two Perfect Squares

The difference of two perfect squares is Tool 4. Understanding this tool requires an understanding of the words “perfect square” and “square root”.

Perfect Square – A perfect square is a number that is created by multiplying a number times itself.

Example: \( 49 \ 64 \ 81 \ 100 \)

\( 2 \times 2 = 4 \) \( 4 \) is a perfect square because it is created by multiplying the number two times itself.

\( 3 \times 3 = 9 \) \( 9 \) is a perfect square because it is created by multiplying the number three times itself.
Perfect square numbers are created by multiplying a number times itself. Perfect square variables simply have exponents that are even.

**Square Root** – A square root is the number that is multiplied times itself to produce a perfect square.

<table>
<thead>
<tr>
<th>Number</th>
<th>...is a square root because...</th>
<th>Multiply</th>
<th>Perfect Square</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>...is a square root because...</td>
<td>7 * 7</td>
<td>49</td>
</tr>
<tr>
<td>5</td>
<td>...is a square root because...</td>
<td>5 * 5</td>
<td>25</td>
</tr>
<tr>
<td>x</td>
<td>...is a square root because...</td>
<td>x * x</td>
<td>x²</td>
</tr>
<tr>
<td>x³</td>
<td>...is a square root because...</td>
<td>x³ * x³</td>
<td>x⁶</td>
</tr>
</tbody>
</table>

***Remember*** To find the square root of a variable, simply divide the exponent by two. This implies that the exponents must be even. Only even numbers are divisible by two.

**Examples:**

\[ x^6, 25x^2y^4, 9z^4, x^4 \]

<table>
<thead>
<tr>
<th>Number</th>
<th>...is a square root because...</th>
<th>Multiply</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>...is a square root because...</td>
<td>7 * 7</td>
</tr>
<tr>
<td>5</td>
<td>...is a square root because...</td>
<td>5 * 5</td>
</tr>
<tr>
<td>x</td>
<td>...is a square root because...</td>
<td>x * x</td>
</tr>
<tr>
<td>x³</td>
<td>...is a square root because...</td>
<td>x³ * x³</td>
</tr>
</tbody>
</table>
When we find the difference of two perfect squares, we must first recognize a perfect square. **Recognition is the key!!!**

**Example:**

\[9x^2 - y^2\] \[25 - z^2\]

\[36x^4z^2 - 49\] \[36(x + 7y)^2 - 49(x + 7y)^2\]

All problems that qualify to be called the difference of two perfect squares, (difference of squares, for short) must have only two terms. **These terms must be separated by a minus sign. The sum of squares cannot be factored.** When we see that the problem is difference of two perfect squares, we must treat them all exactly the same way.

We make two sets of parenthesis: **Step 1** \( (\quad)(\quad) \)

We put a plus sign in one and a minus sign in the other set of parenthesis. The order does not matter.

**Example:** \[25x^2 - y^4\]

**Step 2** \( (\quad +\quad)(\quad -\quad) \)

We find the square root of the first term and write it as the first term in both parenthesis.

**Step 3** \( (5x +\quad)(5x -\quad) \)

We then write the square root of the second term as the second term in both parenthesis.

**Step 4** \( (5x + y^4)(5x - y^4) \)

**Examples:**

\[25y^2 - 36 = (5y + 6)(5y - 6)\]

\[16y^2z^2 - 49 = (4yz + 7)(4yz - 7)\]

**Tool 4**

**The Sum & Difference of Two Perfect Cubes**

**Perfect cube** – A perfect cube is a number that is obtained by multiplying a number times itself three times.
Example: 1 8 27 64 125 216 343

**Cube root:** A cube root is the number that is multiplied times itself three times itself to create a perfect cube.

**Examples:**

- The numbers: 2 is the cube root of 8.
- 3 is the cube root of 27.
- $3 \times 3 \times 3 = 27$ so the cube root of 27 is 3.
- $4 \times 4 \times 4 = 64$ so the cube root of 64 is 4.

Like the exponents of the variables of perfect squares must be divisible by two, the exponents of the variables of perfect cubes must be divisible by three. This implies that the following variables are perfect cubes.

**Examples:**

- $x^3$
- $x^{15}$
- $y^3z^9$

The following terms are perfect cubes:

**Examples:**

- $27x^3$
- $8y^3v^9$
- $64x^3m^6$

***Remember*** The numerical coefficients (the number that is multiplied by the variable), must meet a different requirement than a variable to be called a perfect cube. The numerical coefficient must have a cube root and all variables must have exponents that are divisible by three.

When we factor the difference of two perfect cubes, we must make two sets of parenthesis. They must be of different sizes because a two termed polynomial (binomial) will go in the first set and the second set will house a three termed polynomial (trinomial).

**Example:**

$8x^3 - 27$

Because of the minus sign and the fact that both terms are perfect cube, this is going to be the difference of two perfect cubes. **Recognition is the key!!!**

**Step 1**

Two sets of parenthesis

( ) ( )

The parenthesis must not be of equal size. One must house a binomial—a two termed polynomial, and the other must house trinomial—a three termed polynomial.

**Step 2**

Because this is the difference of two perfect cubes, the signs that must go in the parenthesis must be very specific. The terms of the binomial are separated by a minus sign and the terms of the trinomial are separated by two plus signs.

( - )( + + )
Step 3 \[ 8x^3 - 27 = (2x - 3)( + + ) \]

The rule says that you must write the cube root of the first as the first and the cube root of the second as the second. Since the cube root of the first term is \(2x\), it must be written as the first term of the binomial. The cube root of the second term, which is 3, must be written as the second term in the binomial.

To fill the three spaces of the trinomial, you will not use the original problem. You must ignore the expression \(8x^3 - 27\). Instead, you must take the first term \(2x\) and square it. Squaring it means squaring each factor in the term. \(2x\) then becomes \(4x^2\). \(4x^2\) becomes the first term in the trinomial.

Step 4 \[ 8x^3 - 27 = (2x - 3)(4x^2 + + ) \]

To get the last term in the trinomial, you must square the last term in the binomial. Since the last term in the binomial is 3, when you square it, it becomes 9. This implies that 9 becomes the last term in the trinomial.

\[ 8x^3 - 27 = (2x - 3)(4x^2 + 9) \]

Finally, to get the middle term of the trinomial, you must ignore the signs in the middle two terms in the binomial and just multiply them together. \(2x\) times 3 is \(6x\).

Thus, \[ 8x^3 - 27 = (2x - 3)(4x^2 + 6x + 9) \]

***Remember*** Since the signs were already set at the beginning of the problem, you will ignore the signs of the terms in the binomial and just multiply the terms. We say that we just take their absolute values and then multiply them.

**Formulas:**

\[
\text{Difference of Two Perfect Cubes} \\
(F^3 - L^3) = (F^2 + FL + L^2)
\]

\[
\text{Sum of Two Perfect Cubes} \\
(F^3 + L^3) = (F^2 - FL + L^2)
\]